

Short-term Performance Limits of MIMO Systems with Side Information at the Transmitter

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Abstract

The fundamental performance limits of space-time block code (STBC) designs when perfect channel information is available at the transmitter (CSIT) are studied in this report. With CSIT, the transmitter can perform various techniques such as rate adaption, power allocation, or beamforming. Previously, the exploration of these fundamental results assumed long-term constraints, for example, channel codes can have infinite decoding delay, and power or rate is normalized over infinite channel-uses. With long-term constraints, the transmitter can operate at the rate lower than the instantaneous mutual information and error-free transmission can be supported. In this report, we focus on the performance limits of short-term behavior for STBC systems. We assume that the system has *block power constraint*, *block rate constraint*, and *finite decoding delay*. With these constraints, although the transmitter can perform rate adaption, power control, or beamforming, we show that decoding-error is unavoidable. In the high SNR regime, the diversity gain is upperbounded by the product of the number of transmit antennas, receive antennas, and independent fading block channels that messages spread over. In other words, *fading cannot be completely combatted with short-term constraints*. The proof is based on a sphere-packing argument.

I. INTRODUCTION

There are much interest in the research of side information at the transmitter for multi-input multi-output (MIMO) communication systems in fading channels. Various techniques have been proposed to improve system performance using channel information at the transmitter (CSIT): e.g., beamforming and precoder designs, power allocation and rate adaption methods. The fundamental performance limits of these techniques have been studied extensively. In terms of decoding error probability, the performance limits can be categorized as:

- 1) *Full spatial-and-temporal diversity*: The receiver observes decoding errors. The error decays as a polynomial function of the signal-to-noise ratio (SNR) in the high SNR region. The exponent is

defined as the *diversity gain*, which is limited by the product of the number of transmit antennas, receive antennas, and independent fading block channels.

- 2) *Infinite-diversity*: The receiver observes decoding errors. The error decays exponentially with SNR, like the decoding error in an additive white Gaussian noise (AWGN) channel. The diversity gain is equal to infinite.
- 3) *Error-free transmission*: There is no decoding error at the receiver.

Different system constraints result in different performance limits. One of the constraints that will affect the performance limit is the *decoding delay*, which is defined as the number of channel-uses that the receiver can wait before decoding messages. In other words, it is the number of channel-uses where each information bit spreads. Regarding to decoding delay, there are long-term power and rate constraints as well as short-term power and rate constraints.

Long-term constraints are determined by averaging over all the channel states. A system employing rate adaption with long-term rate constraint, fixed power, and finite decoding delay, can achieve infinite diversity in fading channels [1]. The long-term rate constraint can be written as

$$\int R(\mathbf{h})f(\mathbf{h})d\mathbf{h} \leq R,$$

where the instantaneous transmission rate $R(\mathbf{h})$ depends on the equivalent channel vector \mathbf{h} , and $f(\mathbf{h})$ and R denote the probability density function (PDF) of \mathbf{h} and average transmission rate, respectively. When power allocation with long-term constraint is used for a system with infinite decoding delay, error-free transmission can be achieved for fixed rate. The notion of *delay-limited capacity* is defined as the transmission rate that the system can reliably support for all channel realizations under long-term power allocation at the transmitter [2]. The long-term power constraint can be written as

$$\int P(\mathbf{h})f(\mathbf{h})d\mathbf{h} \leq P,$$

where the transmit power $P(\mathbf{h})$ is a function of the equivalent channel vector \mathbf{h} , and P denotes the average power. It is shown that for single-input single-output (SISO) systems, the delay-limited capacity is zero, while for MIMO systems, there is a nonzero delay-limited capacity.

The long-term constraints are impractical assumptions for real-world implementation. Some applications, for example, video transmission, are delay sensitive and require finite decoding delays. In addition, infinite-length codewords can lead to extremely high decoding complexity. A power allocation with the long-term

constraint may result in an infinite peak power, which is intolerable for electronic devices. Rate adaption with the long-term constraint usually cuts off systems when the receive SNR is low. Nevertheless, delay-sensitive applications need a minimum rate even though channels are in deep fading. For these reasons, power or rate constraint needs to be determined by averaging over a finite number of channel-uses. These constraints define the short-term behavior of communication systems. A mixture of both short-term and long-term constraints have been considered in the literature. For example, a system using long-term power allocation, fixed rate, and finite decoding delay achieves infinite diversity in fading channels [3]. We summarize the diversity performance with respect to different types of constraints and a finite decoding delay in Table I.

TABLE I
DIVERSITY PERFORMANCE FOR MIMO SYSTEMS WITH FINITE DECODING DELAY

Diversity gain	Long-term power constraint	Short-term power constraint
Long-term rate constraint	infinite	infinite [1]
Short-term rate constraint	infinite [3]	unknown

Note that short-term constraints are special cases of long-term constraints. The performance achievable under short-term constraints is achievable under long-term constraints as well. In Table I, the results under both long-term power and rate constraints are straightforward extensions from the results in [1], [3]. Conversely, the results under long-term constraints cannot be used for a system with short-term constraints. To the best of our knowledge, there is no rigorous results under both short-term power and rate constraints.

This report aims at exploring the performance limits of transmitter controls, for example, beamforming, power allocation, rate adaption in conjunction with space-time block code (STBC) designs, under the assumptions of perfect CSIT, finite decoding delay, and short-term power and rate constraints. To allow power allocation and rate adaption within the scope of short-term constraints, we introduce *transmission delay*, the number of channel-uses where power and rate are constrained. The transmission delay is the sum of all decoding delays for STBCs. Assume that a system needs to transmit RT bits of information using L block codes, each with a decoding delay constraint of D_l ($l = 1, \dots, L$), where $T = \sum_{l=1:L} D_l$, denotes the transmission delay, and R denotes the average rate constraint. The transmission for T channel-uses has a *block power constraint* of PT , where P denotes the average power constraint. We assume that the channel information of all T channel-uses is given noncausally to the transmitter. Then, rate adaption or power allocation can be conducted within the scope of the transmission delay. For example, the transmitter is aware that the Frobenius norm of the channel matrix in the first $T/2$ channel-uses is higher than that in the second

$T/2$ channel-uses. Then, a rate adaption and power allocation scheme can send all RT information bits in the first $T/2$ channel-uses using a power equal to $2P$. In the second $T/2$ channel-uses, the transmitter keeps silent.

Note that any transmitter control scheme with short-term constraints on T channel-uses can be viewed as a realization of concatenated STBC with fixed-rate constraint R bits/channel-use, sum-power constraint PT , and decoding delay constraint T channel-uses. The performance limits of the concatenated block code can be applied directly to any transmitter control scheme with short-term constraints. Therefore, we study the performance limits of fixed-rate STBC designs with sum-power and decoding delay constraints when CSIT is available.

For fixed-rate and finite decoding-delay designs, there are two approaches in AWGN channels. Gallager uses the *random coding argument* to show that the decoding error probability drops exponentially with the code length [4]. The random codes show an achievable performance of error probability, which implies the existence of a good code outperforming it. Thus, an upperbound on the achievable error probability is provided by the random codes. Extension of this approach to MIMO systems can be found in [5]. The other approach uses *sphere-packing argument*. It models code-design as a sphere-packing problem. The converse of Shannon capacity can be shown using this argument [6]. It can be extended to MIMO systems with no CSIT to provide a lowerbound on the codeword decoding error probability for any STBC design [7].

To find the fundemantal performance limits of MIMO systems with CSIT, we take the sphere-packing approach to calculate a lowerbound on the codeword decoding error probability given short-term constraints. The main contribution of this report can be summerized as follows:

- 1) For $M \times N$ MIMO systems where each message is encoded over K independent fading blocks, we show that the maximum diversity is MNK under short-term constraints even though the transmitter has perfect CSIT. The result is the same as the scenario when channel information is not available at the transmitter.
- 2) Although [7] assumes a finite decoding delay, the authors assume sphere-hardening at the receiver [6], which implies that the channel codes have infinite-length. Thus, the proposed bound in [7] is not a strict lowerbound on decoding error probability. In this report, we avoid using sphere-hardening arguments. Thus, the results in this report rely completely on finite decoding delays.

We do not claim that our lowerbound is achievable. However, the resulting diversity can be achieved using STBCs. The negative result shows what the system cannot achieve, in other words, the performance limits of STBC design.

The rest of this report is organized as follows. In Section II, we explain the system model. Section III provides a lower bound on error probability. In Section IV, we show the simulated performance of the derived lowerbounds. Conclusions are provided in Section V.

Notation: For a matrix \mathbf{A} , let \mathbf{A}^* , $\text{tr}(\mathbf{A})$, and $\|\mathbf{A}\|$ denote its Hermitian, trace, and Frobenius norm, respectively. We define $\mathcal{CN}(0, 1)$ as circularly symmetrical complex Gaussian distribution with zero mean and unit variance.

II. SYSTEM MODEL

Consider an $M \times N$ MIMO system with M transmit antennas and N receive antennas. The channel coefficients from the transmitter to the receiver are modeled as an independent and identically distributed (i. i. d.) $\mathcal{CN}(0, 1)$ Gaussian random variable. The channels are assumed to be block fading, i.e., the fading coefficients remain fixed for a constant number of channel-uses and change independently from one block to another. We call the interval under which channel coefficients remain unchanged the *block length* of the code. We assume that the block length is L channel-uses.

We consider STBC designs with a maximum usage of K independent channel blocks. In other words, the system has a *decoding delay constraint* of K blocks, or equivalently $T = KL$ channel-uses. The input-output relationship can be described by

$$\mathbf{Y}_k = \mathbf{H}_k \mathbf{X}_k + \mathbf{N}_k, k = 0, \dots, K-1, \quad (1)$$

where \mathbf{Y}_k , \mathbf{H}_k , \mathbf{X}_k , and \mathbf{N}_k denote $N \times L$ receive signal matrix, $N \times M$ channel matrix, $M \times L$ transmit signal matrix, and $N \times L$ AWGN noise matrix, respectively. Each entry of \mathbf{N}_k is i. i. d. $\mathcal{CN}(0, 1)$ Gaussian distributed. The subscript k denotes the index of the block. Let

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_0 \\ \vdots \\ \mathbf{Y}_{K-1} \end{bmatrix}, \mathbf{X} = \begin{bmatrix} \mathbf{X}_0 \\ \vdots \\ \mathbf{X}_{K-1} \end{bmatrix}, \mathbf{N} = \begin{bmatrix} \mathbf{N}_0 \\ \vdots \\ \mathbf{N}_{K-1} \end{bmatrix}, \mathbf{H} = \begin{bmatrix} \mathbf{H}_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{H}_{K-1} \end{bmatrix}. \quad (2)$$

The system equation in (1) can be combined as

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{N}. \quad (3)$$

The transmitter needs to send a set of 2^m messages $\mathcal{M} = \{m_i\}$, where the subscript i is used to represent the message index. A set of 2^m codewords \mathcal{X}_c is generated, where each codeword $\mathcal{X}_i \in \mathcal{X}_c$ has dimension $KM \times L$. For message m_i , the codeword \mathcal{X}_i is selected to be transmitted over the equivalent system in (3). Since 2^m messages are sent in KL channel-uses, the transmission rate can be computed as $R = \frac{m}{KL}$ bits/channel-use. We assume that the equivalent channel matrix \mathbf{H} is noncausally known at the transmitter when designing \mathcal{X}_c . Note that this CSIT assumption is stronger than the causal CSIT assumption, i.e., the transmitter can only know the past and current channels, but not the future channels. The negative results in this report is hence applicable to the scenario of causal CSIT. With CSIT, each codeword \mathcal{X}_i can be a function of \mathbf{H} , i.e. $\mathcal{X}_i(\mathbf{H})$. For simplicity, we use \mathcal{X}_i instead of $\mathcal{X}_i(\mathbf{H})$ throughout the rest of this report. Moreover, we have a *block power constraint* for each codeword \mathcal{X}_i . Mathematically, it can be expressed as $\text{tr}\{\mathcal{X}_i \mathcal{X}_i^*\} \leq KLP$ for $i = 1, \dots, 2^m$.

The receiver is assumed to have perfect channel information, and decodes after receiving the entire block \mathbf{Y} . The maximum-likelihood (ML) method is used to recover transmitted message i as,

$$\hat{i} = \arg \max_{i, \mathcal{X}_i \in \mathcal{X}_c} P(\mathbf{Y} | \mathbf{X} = \mathcal{X}_i, \mathbf{H}), i = 1, \dots, 2^m. \quad (4)$$

A decoding error occurs when $\hat{i} \neq i$. Then, the probability of average codeword decoding error can be defined as

$$P_E = \mathbb{E}_{\mathbf{H}, \mathbf{X}} P(\hat{i} \neq i | \mathbf{X}, \mathbf{H}), \quad (5)$$

where the expectation is taken over all channel realizations of \mathbf{H} and codewords in \mathcal{X}_c . In this report, we aim at finding a lowerbound on P_E for any STBC design with CSIT.

III. A LOWERBOUND ON P_E

In this section, we explain the sphere-packing approach to obtain the lowerbound on P_E . First, a geometrical interpretation is introduced to provide some notations. Then, we tackle the problem relying on these notations.

A geometrical interpretation of the system is described in Fig. 1. Each transmit codeword \mathcal{X}_i can be interpreted as a point in \mathbb{C}^{MLK} . With block power constraint, each codeword can only locate inside a hypersphere. The equivalent channel \mathbf{H} transforms \mathbf{X} into a point in the receive signal space V , which has dimension \mathbb{C}^{NLK} . The transformation includes rotation and scale of the transmit signal space. To illustrate the effect of transformation, the hypersphere in the transmit signal space is transformed into a hyper-ellipsoid

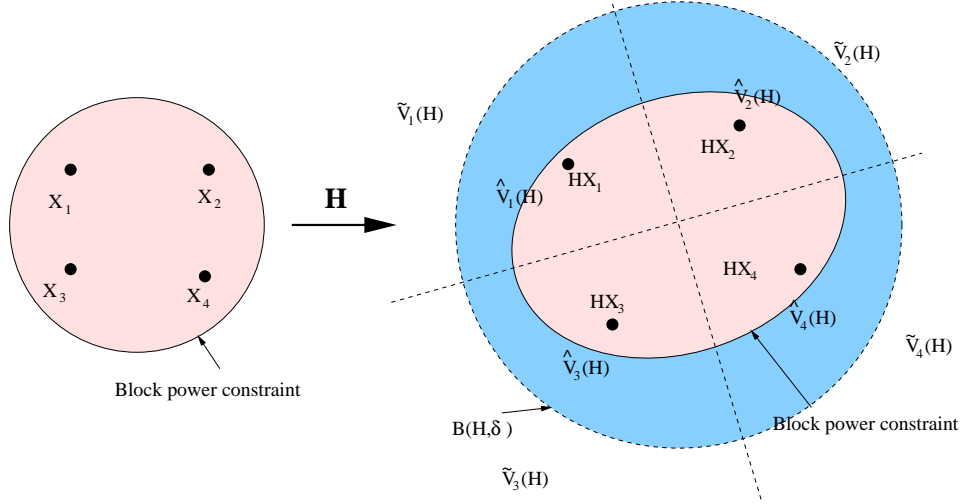


Fig. 1. Geometric interpretation. The transmitter designs a set of four codewords with block power constraint (Left diagram). The receiver decodes by partitioning space into disjoint Voronoi Regions (Right diagram).

in the receive signal space in Fig. 1. The ML decoding method can be equivalently described by partitioning the receive signal space V into disjoint *Voronoi regions*. Let the Voronoi partition be $V = \bigcup_{i=1, \dots, 2^m} V_i(\mathbf{H})$, where $V_i(\mathbf{H})$ denotes the Voronoi region corresponding to \mathcal{X}_i . A Voronoi region $V_i(\mathbf{H})$ is defined based on distance metrics. For any point $\mathbf{Y} \in V_i(\mathbf{H})$, the distance $\|\mathbf{Y} - \mathbf{H}\mathcal{X}_i\|$ is smaller than that of $\|\mathbf{Y} - \mathbf{H}\mathcal{X}_j\|$ for all $j \neq i$. Then, an event of decoding error can be equivalently interpreted as \mathbf{Y} outside the Voronoi region, i.e., $\mathbf{Y} \notin V_i(\mathbf{H})$ given \mathcal{X}_i is sent. Therefore, using the geometrical interpretation, P_E in (5) can be equivalently written as

$$P_E = \mathbb{E}_{\mathbf{H}} P_{E|\mathbf{H}} = \mathbb{E}_{\mathbf{H}} \sum_i P(\mathbf{Y} \notin V_i(\mathbf{H}) | \mathbf{X} = \mathcal{X}_i, \mathbf{H}) P(\mathbf{X} = \mathcal{X}_i), \quad (6)$$

where $P_{E|\mathbf{H}}$ denotes decoding error probability given the channel matrix \mathbf{H} .

We define some regions in the receive signal space V to lowerbound P_E . A hypersphere is defined as

$$B(\mathbf{H}, \delta) = \left\{ \text{tr}(\mathbf{Y}\mathbf{Y}^*) \leq \left(\sqrt{PLK \text{tr}(\mathbf{H}\mathbf{H}^*)} + \sqrt{NLK\delta} \right)^2 \right\}, \quad (7)$$

where δ is any positive parameter to control the radius of the hypersphere. The choice of δ will be discussed later. The hypersphere defines a region that the receive signal resides with a high probability due to block power constraint. Further, we denote $\overline{B(\mathbf{H}, \delta)}$ as the region outside the hypersphere $B(\mathbf{H}, \delta)$. Thus, the whole receive signal space can be partitioned into the part inside hypersphere and the part outside hypersphere, i.e., $V = B(\mathbf{H}, \delta) \cup \overline{B(\mathbf{H}, \delta)}$. Since the volume of $V_i(\mathbf{H})$ may be unbounded, we further partition each $V_i(\mathbf{H})$

into two parts with reference to $B(\mathbf{H}, \delta)$. We define

$$\hat{V}_i(\mathbf{H}) = V_i(\mathbf{H}) \cap B(\mathbf{H}, \delta), \quad \tilde{V}_i(\mathbf{H}) = V_i(\mathbf{H}) \cap \overline{B(\mathbf{H}, \delta)}, \quad (8)$$

where $\hat{V}_i(\mathbf{H})$ is the part of $V_i(\mathbf{H})$ inside the hypersphere $B(\mathbf{H}, \delta)$ and $\tilde{V}_i(\mathbf{H})$ is the part outside of the hypersphere. Fig. 1 illustrates the partition of receive signal space for a set of four codewords, i.e., $m = 2$.

In what follows, we obtain a lowerbound on (6). For simplicity, we can omit \mathbf{H} in the notations used in (6), (7), and (8). It follows

$$\begin{aligned} P_{E|\mathbf{H}} &= \sum_i P(\mathbf{Y} \notin V_i | \mathbf{X} = \mathcal{X}_i) P(\mathbf{X} = \mathcal{X}_i) \\ &= \sum_i P(\mathbf{Y} \notin \hat{V}_i | \mathbf{X} = \mathcal{X}_i) P(\mathbf{X} = \mathcal{X}_i) - \sum_i P(\mathbf{Y} \in \tilde{V}_i | \mathbf{X} = \mathcal{X}_i) P(\mathbf{X} = \mathcal{X}_i) \\ &> \sum_i P(\mathbf{Y} \notin \hat{V}_i | \mathbf{X} = \mathcal{X}_i) P(\mathbf{X} = \mathcal{X}_i) - \sum_i P(\mathbf{Y} \in \overline{B(\delta)} | \mathbf{X} = \mathcal{X}_i) P(\mathbf{X} = \mathcal{X}_i) \\ &= \underbrace{\sum_i P(\mathbf{Y} \notin \hat{V}_i | \mathbf{X} = \mathcal{X}_i) P(\mathbf{X} = \mathcal{X}_i)}_A - P(\mathbf{Y} \in \overline{B(\delta)}), \end{aligned} \quad (9)$$

where we have Line 2 since \tilde{V}_i and \hat{V}_i are disjoint sets and $\tilde{V}_i \cup \hat{V}_i = V_i$; the inequality in Line 3 is true since \tilde{V}_i is included in $\overline{B(\delta)}$. The following two lemmas are needed to provide a lowerbound on P_E .

Lemma 1: Let $S(r_i)$ be an (nLK) -hypersphere centered at $\mathbf{H}\mathcal{X}_i$ with a radius of r_i . The radius r_i is selected such that $S(r_i)$ and \hat{V}_i have the same volume. Substituting \hat{V}_i with $S(r_i)$ in $P(\mathbf{Y} \notin \hat{V}_i | \mathbf{X} = \mathcal{X}_i)$, we have

$$P(\mathbf{Y} \notin \hat{V}_i | \mathbf{X} = \mathcal{X}_i) \geq P(\mathbf{Y} \notin S(r_i) | \mathbf{X} = \mathcal{X}_i)$$

Proof: See [8]. ■

Intuitively, this lemma can be explained using Fig. 2. The PDF of \mathbf{Y} given \mathcal{X}_i depends only on the radius r_i . For any point in Region III, its PDF is higher than that of any point in Region II. Then, the probability of the receive signal in Region III is higher than that in Region II. As a result, the probability of \mathbf{Y} being inside the sphere $S(r_i)$ is higher than that inside \hat{V}_i . Conversely, it is less likely for \mathbf{Y} to be outside $S(r_i)$ than \hat{V}_i .

Lemma 2: The probability of \mathbf{Y} to be outside of the hypersphere $B(\delta)$ is upperbounded by

$$P(\mathbf{Y} \in \overline{B(\delta)}) \leq \frac{\Gamma(NLK, NLK\delta)}{\Gamma(NLK)},$$

where $\Gamma(n, x)$ denotes the incomplete Gamma function, i.e., $\Gamma(n, x) = \int_x^{+\infty} t^{n-1} e^{-t} dt$.

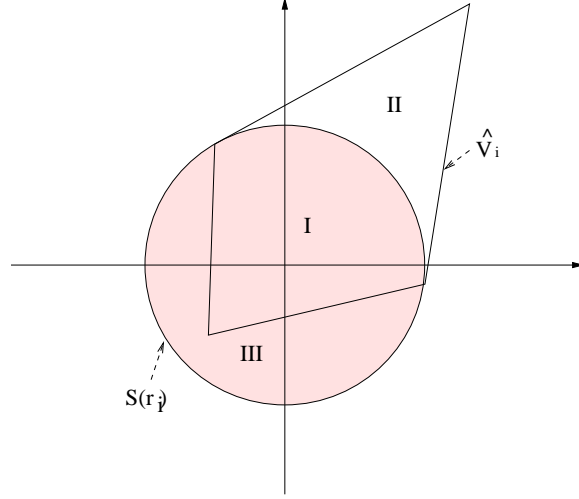


Fig. 2. Lemma 1: Lowerbounds using sphere as Voronoi Region.

Proof: From (3), \mathbf{Y} is the sum of $\mathbf{H}\mathbf{X}$ and \mathbf{N} . With a block power constraint, the Frobenius norm of the first term can be bounded as $\text{tr}(\mathbf{X}^*\mathbf{H}^*\mathbf{H}\mathbf{X}) \leq \text{tr}(\mathbf{X}^*\mathbf{X})\text{tr}(\mathbf{H}^*\mathbf{H}) = KLP\text{tr}(\mathbf{H}^*\mathbf{H})$. Then, we have

$$\mathbb{P}\left(\text{tr}(\mathbf{X}^*\mathbf{H}^*\mathbf{H}\mathbf{X}) \leq KLP\text{tr}(\mathbf{H}^*\mathbf{H})\right) = 1.$$

From (2), since \mathbf{N} is the equivalent $NK \times L$ noise matrix and each entry is i. i. d. $\mathcal{CN}(0, 1)$ distributed, $\text{tr}(\mathbf{N}\mathbf{N}^*)$ is Chi-square distributed with $2NLK$ degrees of freedom. Then, we can compute

$$\mathbb{P}\left(\text{tr}(\mathbf{N}\mathbf{N}^*) \leq NLK\delta\right) = 1 - \frac{\Gamma(NLK, NLK\delta)}{\Gamma(NLK)}.$$

Since the norm of the sum of two matrices can be upperbounded by the sum of the norms of each matrix, we have $\sqrt{\text{tr}(\mathbf{Y}^*\mathbf{Y})} \leq \sqrt{\text{tr}(\mathbf{X}^*\mathbf{H}^*\mathbf{H}\mathbf{X})} + \sqrt{\text{tr}(\mathbf{N}\mathbf{N}^*)}$. The probability of \mathbf{Y} falling into the hypersphere $B(\delta)$ can be bounded as

$$\begin{aligned} \mathbb{P}(\mathbf{Y} \in B(\delta)) &= \mathbb{P}\left(\text{tr}(\mathbf{Y}^*\mathbf{Y}) \leq \left(\sqrt{KLP\text{tr}(\mathbf{H}^*\mathbf{H})} + \sqrt{NLK\delta}\right)^2\right) \\ &= \mathbb{P}\left(\sqrt{\text{tr}(\mathbf{Y}^*\mathbf{Y})} \leq \sqrt{KLP\text{tr}(\mathbf{H}^*\mathbf{H})} + \sqrt{NLK\delta}\right) \\ &\geq \mathbb{P}\left(\sqrt{\text{tr}(\mathbf{X}^*\mathbf{H}^*\mathbf{H}\mathbf{X})} + \sqrt{\text{tr}(\mathbf{N}\mathbf{N}^*)} \leq \sqrt{KLP\text{tr}(\mathbf{H}^*\mathbf{H})} + \sqrt{NLK\delta}\right) \\ &\geq \mathbb{P}\left(\text{tr}(\mathbf{X}^*\mathbf{H}^*\mathbf{H}\mathbf{X}) \leq KLP\text{tr}(\mathbf{H}^*\mathbf{H})\right) \mathbb{P}\left(\text{tr}(\mathbf{N}\mathbf{N}^*) \leq NLK\delta\right) = 1 - \frac{\Gamma(NLK, NLK\delta)}{\Gamma(NLK)}. \end{aligned}$$

Therefore, the probability that \mathbf{Y} is outside the hypersphere $B(\delta)$ is upperbounded by $\frac{\Gamma(NLK, NLK\delta)}{\Gamma(NLK)}$. ■

Lemma 2 says that with a block power constraint, the receive signal \mathbf{Y} is constrained in the hypersphere $B(\delta)$ with a high probability. Proving Lemmas 1 and 2, we are ready for the following theorem.

Theorem 1: When the messages are equiprobable, i.e., $P(\mathbf{X} = \mathcal{X}_i) = \frac{1}{2^m}$, a lowerbound on P_E can be obtained as¹

$$P_E \geq \frac{P^{NLK} \Gamma(NLK + MNK)}{\Gamma(NLK) \Gamma(MNK)} \int_a^b \frac{\mathcal{X}^{NLK-1}}{(1 + P\mathcal{X})^{MNK+NLK}} d\mathcal{X}, \quad (10)$$

where the bounds of the integral are $a = LK2^{-R/N} \left(\frac{\sqrt{2}}{2^{R/(2N)} - 1} + 1 \right)^2$, $b = \frac{2LK}{(2^{R/(2N)} - 1)^2}$.

Proof: When messages are equiprobable, from Lemma 1, we can further lowerbound the term A in (9) as

$$\begin{aligned} A &= \frac{1}{2^m} \sum_i P(\mathbf{Y} \notin \hat{V}_i | \mathbf{X} = \mathcal{X}_i) \\ &\geq \frac{1}{2^m} \sum_i P(\mathbf{Y} \notin S(r_i) | \mathbf{X} = \mathcal{X}_i) = \frac{1}{2^m} \sum_i \frac{\Gamma(NLK, r_i^2)}{\Gamma(NLK)}. \end{aligned} \quad (11)$$

Substituting (11) and the result of Lemma 2 into (9), we can lowerbound $P_{E|H}$ as

$$P_{E|H} \geq \frac{1}{2^m} \sum_i \frac{\Gamma(NLK, r_i^2)}{\Gamma(NLK)} - \frac{\Gamma(NLK, NLK\delta)}{\Gamma(NLK)}. \quad (12)$$

In what follows, we further lowerbound the RHS of (12) to help analyze diversity. Note that \hat{V}_i is inside the hypersphere $B(\delta)$ and \hat{V}_i has the same volume as $S(r_i)$. There is an additional constraint on the sum of the volumes of $S(r_i)$, i.e., $\text{Vol}(B(\delta)) = \sum_i \text{Vol}(S(r_i))$. Thus, an optimization problem can be formulated as

$$\begin{aligned} \min_{r_i} \quad & \frac{1}{2^m} \sum_i \frac{\Gamma(NLK, r_i^2)}{\Gamma(NLK)} - \frac{\Gamma(NLK, NLK\delta)}{\Gamma(NLK)} \\ \text{s.t.} \quad & \sum_i \text{Vol}(S(r_i)) = \text{Vol}(B(\delta)), \\ & \delta > 0. \end{aligned} \quad (13)$$

Since δ is a constant, the second term in the objective function can be ignored. This minimization problem is known as *sphere-packing*. The solution is obtained when each hypersphere $S(r_i)$ has equal radius, i.e., $r_i = r_o(\delta)$ for $i = 1, \dots, 2^m$ [7]. Since the volume of a hypersphere in \mathbb{C}^n is $\text{Vol} = R_n r^{2n}$ where R_n is a constant depending on dimensions and r is the radius, from the constraint, we have

$$2^m R_n r_o(\delta)^{2NLK} = R_n \left(\sqrt{PLK \text{tr}(\mathbf{H}\mathbf{H}^*)} + \sqrt{NLK\delta} \right)^{2NLK}.$$

After dividing both sides of the above equation by $2^m R_n$ and taking NLK th root, we have

$$r_o(\delta)^2 = NLK 2^{-\frac{R}{N}} \left(\sqrt{\delta} + \sqrt{\frac{P}{N} \text{tr}(\mathbf{H}\mathbf{H}^*)} \right)^2, \quad (14)$$

¹When messages are not uniformly distributed, we can use $\min_i P(\mathbf{X} = \mathcal{X}_i)$ to lowerbound $P(\mathbf{X} = \mathcal{X}_i)$. It is straightforward to extend the results to the case of non-uniform messages.

where $R = \frac{m}{LK}$, denoting the bit-rate per channel-uses of the STBC designs. Replacing (14) into (12), we have a lowerbound on $P_{E|H}$ as

$$P_{E|H} \geq \frac{\Gamma(NLK, r_o(\delta)^2)}{\Gamma(NLK)} - \frac{\Gamma(NLK, NLK\delta)}{\Gamma(NLK)}. \quad (15)$$

In what follows, we discuss the choice of δ . In [7], δ is chosen to be one, and the probability of \mathbf{Y} to be outside the hypersphere, i.e., the second term in Line 4 of (9), is not taken into account. Therefore, the lowerbound obtained in [7] is not a tight lowerbound. In this report, we formulate the lowerbound considering both the events when \mathbf{Y} is inside and outside the hypersphere. From (14), the radius of each hypersphere r_o depends on δ , and for any positive δ , the RHS of (15) provides a new lowerbound. To find an explicit lowerbound on P_E , we choose a δ that results in a positive number on the RHS of (15)². Note that the first and second terms in (15) are both incomplete Gamma functions with NLK degrees of freedom. To have a positive lowerbound, we need

$$r_o(\delta)^2 < NLK\delta. \quad (16)$$

Substituting (14) into (16), we have

$$2^{-\frac{R}{N}} \left(\sqrt{\delta} + \sqrt{\frac{P}{N} \text{tr}(\mathbf{H}\mathbf{H}^*)} \right)^2 < \delta$$

$$\frac{\delta}{\frac{P}{N} \text{tr}(\mathbf{H}\mathbf{H}^*)} > \frac{1}{\left(2^{\frac{R}{N}} - 1\right)^2}.$$

Then, we let $\delta = \frac{2P}{N(2^{\frac{R}{N}} - 1)^2} \text{tr}(\mathbf{H}\mathbf{H}^*)$. We can expand the RHS of (15) into an integral as

$$P_{E|H} \geq \frac{1}{\Gamma(NLK)} \int_{r_o^2}^{NLK\delta} x^{NLK-1} e^{-x} dx$$

$$= \frac{1}{\Gamma(NLK)} \int_{\frac{2PLK}{(2^{R/(2N)}-1)^2} \text{tr}(\mathbf{H}\mathbf{H}^*)}^{\frac{2PLK}{(2^{R/(2N)}-1)^2} \text{tr}(\mathbf{H}\mathbf{H}^*)} x^{NLK-1} e^{-x} dx. \quad (17)$$

Further let $a = LK2^{-R/N} \left(\frac{\sqrt{2}}{2^{R/(2N)}-1} + 1 \right)^2$, $b = \frac{2LK}{(2^{R/(2N)}-1)^2}$, and $h = \text{tr}(\mathbf{H}\mathbf{H}^*)$ to simplify the notations, and integrate (17) over h . From (2), since $\text{tr}(\mathbf{H}\mathbf{H}^*)$ is Chi-square distributed with $2MNK$ degrees of

²Note that the best lowerbound can be found by further maximizing (15) with respect to δ . The optimal δ can be obtained by calculating the derivative of (15) and set the derivative to zero. The resulting equality is nonlinear in δ , and cannot be solved explicitly. Therefore, we cannot find an explicit lowerbound on P_E using the optimal δ .

freedoms, we obtain a lowerbound on P_E as

$$\begin{aligned}
P_E &\geq \frac{1}{\Gamma(NLK)\Gamma(MNK)} \int_0^{+\infty} h^{MNK-1} e^{-h} \int_{aPh}^{bPh} x^{NLK-1} e^{-x} dx dh \\
&= \frac{1}{\Gamma(NLK)\Gamma(MNK)} \int_0^{+\infty} h^{MNK+NLK-1} P^{NLK} e^{-h} \int_a^b \mathcal{X}^{NLK-1} e^{-P\mathcal{X}} d\mathcal{X} dh \\
&= \frac{P^{NLK}}{\Gamma(NLK)\Gamma(MNK)} \int_a^b \mathcal{X}^{NLK-1} d\mathcal{X} \int_0^{+\infty} h^{MNK+NLK-1} e^{-h(1+P\mathcal{X})} dh \\
&= \frac{P^{NLK}\Gamma(NLK+MNK)}{\Gamma(NLK)\Gamma(MNK)} \int_a^b \frac{\mathcal{X}^{NLK-1}}{(1+P\mathcal{X})^{MNK+NLK}} d\mathcal{X}
\end{aligned} \tag{18}$$

In Line 2, we have replaced the variable x with $\mathcal{X}Ph$. This concludes the proof. \blacksquare

The proof of Theorem 1 uses the sphere-packing argument. The diversity performance is given in the following corollary.

Corollary 1: With short-term decoding delay, block power, and block rate constraints, the diversity of any STBC design is upperbounded by MNK . It is independent of whether CSI is available at the transmitter or not.

Proof: The diversity gain is defined as

$$d = -\lim_{P \rightarrow \infty} \frac{\log P_E}{\log P}. \tag{19}$$

Replacing the lowerbound obtained in Theorem 1 into (19) results in an upperbound on diversity gain. Thus, we have

$$\begin{aligned}
d &\leq -\lim_{P \rightarrow \infty} \frac{\log \left(\frac{P^{NLK}\Gamma(NLK+MNK)}{\Gamma(NLK)\Gamma(MNK)} \int_a^b \frac{\mathcal{X}^{NLK-1}}{(1+P\mathcal{X})^{MNK+NLK}} d\mathcal{X} \right)}{\log P} \\
&= -NLK - \lim_{P \rightarrow \infty} \frac{\log \left(\int_a^b \frac{\mathcal{X}^{NLK-1}}{(1+P\mathcal{X})^{MNK+NLK}} d\mathcal{X} \right)}{\log P} \\
&\leq -NLK - \lim_{P \rightarrow \infty} \frac{\log \left(\frac{1}{(1+Pb)^{MNK+NLK}} \int_a^b \mathcal{X}^{NLK-1} d\mathcal{X} \right)}{\log P} \\
&= -NLK - \lim_{P \rightarrow \infty} \frac{\log \left(\frac{1}{(1+Pb)^{MNK+NLK}} \right)}{\log P} = MNK.
\end{aligned}$$

In Line 3, we lowerbound the integrand $\frac{\mathcal{X}^{NLK-1}}{(1+P\mathcal{X})^{MNK+NLK}}$ by $\frac{\mathcal{X}^{NLK-1}}{(1+Pb)^{MNK+NLK}}$; In Line 4, we ignore the integral because for a fixed-rate R , the bounds a and b are independent of power P .

Note that in all previous lowerbounding steps, the inequalities are independent of CSIT. Therefore, the derived lowerbound is independent of CSIT and is applicable to both CSIT and no CSIT scenarios. \blacksquare

The lowerbounding techniques used in the proof of Corollary 1 motivate two explicit lowerbounds where

no integral is involved. From (18), we have the following two bounds.

$$\begin{aligned} \text{Bound 1 : } P_E &> \frac{\Gamma(NLK + MNK)}{\Gamma(NLK)\Gamma(MNK)} \frac{P^{NLK}}{(1 + Pb)^{MNK+NLK}} \int_a^b \mathcal{X}^{NLK-1} d\mathcal{X} \\ &= \frac{\Gamma(NLK + MNK)}{\Gamma(NLK + 1)\Gamma(MNK)} \frac{P^{NLK}}{(1 + Pb)^{MNK+NLK}} (b^{NLK} - a^{MLK}). \end{aligned} \quad (20)$$

$$\begin{aligned} \text{Bound 2 : } P_E &> \frac{\Gamma(NLK + MNK)}{\Gamma(NLK)\Gamma(MNK)} a^{NLK-1} P^{NLK} \int_a^b \frac{1}{(1 + P\mathcal{X})^{MNK+NLK}} d\mathcal{X} \\ &= \frac{\Gamma(NLK + MNK - 1)}{\Gamma(NLK)\Gamma(MNK)} a^{NLK-1} P^{NLK} ((1 + Pa)^{-NLK-MNK+1} - (1 + Pb)^{-NLK-MLK+1}). \end{aligned} \quad (21)$$

IV. NUMERICAL RESULTS

In this section, we demonstrate the results of our analysis and compare it with the results of beamforming schemes using singular-value decomposition (SVD). We consider two systems: System A with parameters $M = 2, N = 1$; System B with parameters $M = 2, N = 2; L = 1, K = 1$ for both Systems A and B. For these two simple cases, we assume no channel coding, and the decoding error probability in (5) is equivalent to the symbol error rate (SER). We apply BPSK, QPSK, 8QAM modulation for the SVD schemes, whose corresponding rates are $R = 1, 2, 3$ bits per channel-uses, respectively. We plot the two strict lowerbounds in (20) and (21).

Figs. 3 and 4 compare the SER of SVD schemes with derived lowerbounds in Systems A and B, respectively. Although the lowerbounds are loose in terms of array gain, the diversity gain can be observed to be tight.

V. CONCLUSION

We have obtained a negative result for STBC systems using transmitter control with short-term constraints on decoding delay, block power, and block rates. The analysis shows that fading cannot be completely combatted with short-term constraints. The diversity is upperbounded by the product of the numbers of transmit antennas, receive antennas, and the independent fading block channels that messages span over.

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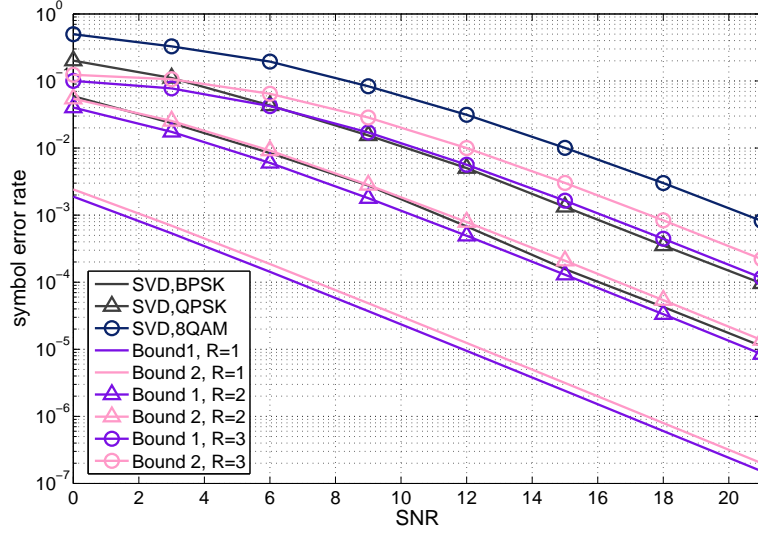


Fig. 3. SER in a 2×1 system with one channel-uses.

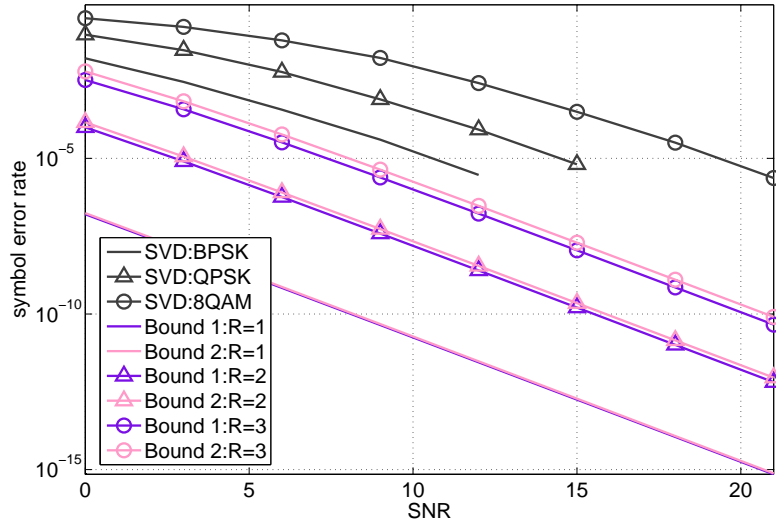


Fig. 4. SER in a 2×2 system with one channel-uses.

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